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Review Article

Interpolation-Collacation Method for the Determination of Heat Coduction through a Large Flat Steel Plate

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Abstract

A new continuous numerical approach based on the approximation of polynomials is here proposed for solving the equation arising from heat transfer along a large flat steel plate subject to initial and boundary conditions. The method results from discretization of the heat equation which leads to the production of a system of algebraic equations. By solving the system of algebraic equations we obtain the problem approximate solutions.

Introduction

The development of continuous numerical techniques for solving heat conduction equation in science and engineering subject to initial and boundary conditions is a subject of considerable interest. In this paper, we develop a new continuous numerical method which is based on interpolation and collocation at some point along the coordinates [1-17]. To do this we let $U(x,t)$ represents the temperature at any point in the large flat steel plate. Heat is flowing from one end to another under the influence of the temperature gradient $\partial U/\partial x$. To make a balance of the rate of heat flow in and out of the medium, we consider R for thermal conductivity of the steel, C the heat capacity which we assume constants and ρ the density. Heat flow in the plate is given by

$$-RA\frac{\partial U}{dx}-\left[-RA\left[\frac{\partial U}{dx}+\frac{\partial}{\partial x}\left(\frac{\partial U}{dx}\right)dx\right]\right]=C\rho\left(Adx\right)\frac{\partial U}{dt} \quad (1.0)$$

Where A is the cross section of the flat plate?

Our new method strives to provide solutions to the heat flow eqn. (1.0).

Solution Method

To set up the solution method we select an integer N such that $N>0$. We subdivide the interval $0\le x\le X$ into N equal subintervals with mesh points along space axis given by $x_i=ih,\ i=\frac{1}{\beta}\left(\frac{1}{\beta}\right)N$, where $Nh=X$. Similarly, we reverse the roles of X and t and we select another integer M such that $M>0$. We also subdivide the interval $0\le t\le T$ into M equal subintervals with mesh points along time axis given by $t_j=jk$, $j=\frac{1}{\alpha}\left(\frac{1}{\alpha}\right)M$ where $Mk=T$ and h,k are the mesh sizes along space and time axes respectively. Here, we seek for the approximate solution $\overline{U}(x,t)$ to $U(x,t)$ of the form

$$\overline{U}(x,t)\approx\overline{U}_{p-1}(x,t)=\sum_{r=0}^{p-1}a_r\left[q_r(x,t)+s_r(x,t)+\psi_r(x,t)\right],\ x\in\left[x_i,x_{i+h}\right],t\in\left[t_j,t_{j+k}\right] \quad (2.0)$$

Over $h>0,\ k>0$ mesh sizes, such that $0=x_0<...<x_i<...<x_N,\ 0=t_0<...<t_j<...<t_M$. We denote p to be the sum of interpolation points along the space and time coordinates respectively. That is $p=g+b$, where g is the number of interpolation points along the space coordinate, while b is the number of interpolation points along the time coordinate. The bases functions q_r, s_r , ψ_r, $r=0,1,...,p-1$ are the Taylor's, Legendre's and chebyshev's polynomials which are known, a_r are the constants to be determined. The interpolation values $\overline{U}_{i,j},...,\overline{U}_{i+h-1,j}$ are assumed to have been determined from previous steps, while the method seeks to obtain $\overline{U}_{i+h,j}$ (Odekunle, 2008). Applying the above interpolation conditions on

eqn. (2.0) we obtain

$$\overline{U}_{i+h,j+k}(x,t)=a_0(q_0+s_0+\psi_0)(x_{i+h},t_{j+k})+a_1(q_1+s_1+\psi_1)(x_{i+h},t_{j+k})+...$$
$$+a_{p-1}(q_{p-1}+s_{p-1}+\psi_{p-1})(x_{i+h},t_{j+k}) \quad (2.1)$$

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We let $h = -\frac{1}{\beta}\left(\frac{1}{\beta}\right)\left[g - \left(\frac{2\beta - 1}{\beta}\right)\right]$ arbitrarily and $k = 0$, then, by Cramer's rule,

eqn. (2.1) becomes

$$W\underline{a} = \underline{F}, \qquad \underline{F} = \left(U_{v,j}, U_{v+\frac{1}{\beta},j}, \dots, U_{z,j}\right)^T, \tag{2.2}$$

$$\underline{a} = \left(a_0, \dots, a_{p-1}\right)^T \quad \text{And}$$

$$W = \begin{bmatrix} (q_0 + s_0 + \psi_0)(x_v, t_j), & (q_1 + s_1 + \psi_1)(x_v, t_j), & \dots, (q_{p-1} + s_{p-1} + \psi_{p-1})(x_v, t_j) \\ (q_0 + s_0 + \psi_0)\left(x_{v+\frac{1}{\beta}}, t_j\right), & (q_1 + s_1 + \psi_1)\left(x_{v+\frac{1}{\beta}}, t_j\right), & \dots, (q_{p-1} + s_{p-1} + \psi_{p-1})\left(x_{v+\frac{1}{\beta}}, t_j\right) \\ \dots, & \dots, & \dots \\ (q_0 + s_0 + \psi_0)(x_z, t_j), & (q_1 + s_1 + \psi_1)(x_z, t_j), & \dots, (q_{p-1} + s_{p-1} + \psi_{p-1})(x_z, t_j) \end{bmatrix}$$

where $z = i + g - \left(\frac{2\beta - 1}{\beta}\right)$, $v = i - \frac{1}{\beta}$ and W^{-1} exist Odekunle, 2008).

Hence, from eqn. (2.2), we obtain

$$\underline{a} = \varpi \underline{F}, \qquad \varpi = W^{-1}. \tag{2.3}$$

The vector $\underline{a} = \left(a_0, \dots, a_{p-1}\right)^T$ is now determined in terms of known parameters in $\varpi \underline{F}$. If ϖ_{r+1} is the $(r+1)^{th}$ row of ϖ, then

$$a_r = \underline{\varpi}_{r+1} \underline{F} \tag{2.4}$$

Eqn. (2.4) determines the values of a_r explicitly.

We take first and second derivatives of eqn. (2.0) with respect to x

$$\overline{U}'(x,t) = \sum_{r=0}^{p-1}\left[a_r\left(q_r'(x,t) + s_r'(x,t) + \psi_r'(x,t)\right)\right]$$

$$\overline{U}''(x,t) = \sum_{r=0}^{p-1}\left[a_r\left(q_r''(x,t) + s_r''(x,t) + \psi_r''(x,t)\right)\right] \tag{2.5}$$

Substituting eqn. (2.4) into eqn. (2.5) we obtain

$$\overline{U}''(x,t) = \sum_{r=0}^{p-1}\left[\underline{\varpi}_{r+1}\underline{F}\left(q_r''(x,t) + s_r''(x,t) + \psi_r''(x,t)\right)\right] \tag{2.6}$$

We then reverse the roles of x and t in eqn. (2.1) and arbitrarily set

$k = 0\left(\frac{1}{\alpha}\right)\left[b - \left(\frac{\alpha - 1}{\alpha}\right)\right], h = 0$ then by Cramer's rule eqn. (2.1) becomes

$$Y\underline{a} = \underline{E}, \qquad \underline{E} = \left(U_{i,\eta-\frac{1}{\alpha}}, U_{i,\eta}, \dots, U_{i,\gamma}\right)^T \tag{2.7}$$

$$\underline{a} = \left(a_0, \dots, a_{p-1}\right)^T \qquad \text{and}$$

$$Y = \begin{bmatrix} (q_0 + s_0 + \psi_0)\left(x_i, t_{\eta-\frac{1}{\alpha}}\right), & (q_1 + s_1 + \psi_1)\left(x_i, t_{\eta-\frac{1}{\alpha}}\right), & \dots, (q_{p-1} + s_{p-1} + \psi_{p-1})\left(x_i, t_{\eta-\frac{1}{\alpha}}\right) \\ (q_0 + s_0 + \psi_0)(x_i, t_\eta), & (q_1 + s_1 + \psi_1)(x_i, t_\eta), & \dots, (q_{p-1} + s_{p-1} + \psi_{p-1})(x_i, t_\eta) \\ \dots, & \dots, & \dots, \dots \\ (q_0 + s_0 + \psi_0)(x_i, t_\gamma), & (q_1 + s_1 + \psi_1)(x_i, t_\gamma), & \dots, (q_{p-1} + s_{p-1} + \psi_{p-1})(x_i, t_\gamma) \end{bmatrix}$$

where $\eta = j + \frac{1}{\alpha}$, $\gamma = j + b - \left(\frac{\alpha - 1}{\alpha}\right)$ and Y^{-1} exist (Odekunle, 2008).

Hence, from eqn. (2.7) we obtain

$$\underline{a} = L\underline{E}, \qquad L = Y^{-1}. \tag{2.8}$$

The vector $\underline{a} = \left(a_0, \dots, a_{p-2}\right)^T$ is now determined in terms of known parameters in $L\underline{E}$.

If L_{r+1} is the $(r+1)^{th}$ row of L, then

$$a_r = \underline{L}_{r+1}\underline{E} \tag{2.9}$$

Also, eqn. (2.9) determines the values of a_r clearly. Taking the first derivatives of

eqn. (2.0) with respect to t we obtain

$$\overline{U}'(x,t) = \sum_{r=0}^{p-1}\left[a_r\left(q_r'(x,t) + s_r'(x,t) + \psi_r'(x,t) + \Phi_r'(x,t)\right)\right] \tag{2.10}$$

Substituting eqn. (2.9) into eqn. (2.10) we obtain

$$\overline{U}'(x,t) = \sum_{r=0}^{p-1}\left[\underline{L}_{r+1}\underline{E}\left(q_r'(x,t) + s_r'(x,t) + \psi_r'(x,t)(x,t)\right)\right] \tag{2.11}$$

But by eqn. (1.0) it is obvious that eqn. (2.11) is equal to eqn. (2.6), therefore,

$$\sum_{r=0}^{p-1}\left[\underline{L}_{r+1}\underline{E}\left(q_r'(x,t) + s_r'(x,t) + \psi_r'(x,t)\right)\right]$$

$$-\sum_{r=0}^{p-1}\left[\underline{\varpi}_{r+1}\underline{F}\left(q_r''(x,t) + s_r''(x,t) + \psi_r''(x,t)\right)\right] = 0 \tag{2.12}$$

Collocating eqn. (2.12) at $x = x_i$ and $t = t_j$ produces a new continuous numerical scheme that solves eqn. (1.0) explicitly.

Numerical examples

In this section, we will test the numerical accuracy of the new method by using the new scheme to solve two examples. That is, we compute approximate solutions of equation. (1.0) at each time level. To achieve this, we truncate the polynomials after second degree, and the average is used as the basis function in the computation. The resultant scheme is used to solve the following two problems:

Example 1 Benner and Mena [18-27]

A large flat steel plate is 4cm thick. If the initial temperatures 0^oC within the plate are given as a function of the distance from one face, by the equations

$$U = 100x \quad for \ \ 0 \le x \le 2, \qquad U = 100(4 - x), \quad for \ \ 2 \le x \le 4$$

Find the temperatures as a function of x and t if both faces are maintained at 0^oC. Where k for steel is 0.13, $c = 0.11$, $\rho = 7.8$.

By simplification eqn. (2.0) becomes $A\frac{\partial^2 U}{\partial x^2} = c\rho\frac{\partial U}{\partial t}$. To solve this equation, we subdivide the total thickness into an integral number of spaces. Let us use $\Delta x = 0.80cm$,

$$\frac{k\Delta t}{cp(\Delta x)^2} = \frac{85}{16}, \Rightarrow \frac{0.13 \times \Delta t}{0.11 \times 7.8(0.8)^2}, \quad 2.08\Delta t = 46.68, \ \ \Delta t = 22.44$$

Taking $\beta = 4$, $\alpha = 170$ implies that $v = i - \frac{1}{4}$, $z = i + \frac{1}{4}$ and $\eta = \gamma = j + \frac{1}{170}$. We take two interpolation

Table 1: Calculated temperatures.

t	X=0	X=0.25	X=0.50	X=0.75	X=1.0	X=1.25
0.0	0.0	25	50	75	100	75
0.26	0.0	25	50	75	93.75	75
0.412	0.0	25	50	74.22	89.06	74.22
0.619	0.0	25	49.90	73.05	85.35	73.05
0.825	0.0	24.99	9.68	71.69	82.28	71.69
1.031	0.0	24.95	49.35	70.26	79.62	70.26

points along space coordinate and one interpolation point along time coordinate. Implies $g = 2,\ b = 1,\ \Rightarrow\ p = 3$ and for $i = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots,\ and\ \ j = \frac{1}{170}, \frac{1}{85}, \frac{3}{170}, \ldots,$ implies that $h = -\frac{1}{4}, 0, \frac{1}{4}\quad and\ \ k = 0, \frac{1}{170}$, then the calculated temperatures are as shown in table 1.

Example 2 (Benner and Mena [5])

A large flat steel plate is 2cm thick. If the initial temperatures $0^{o}C$ within the plate are given as a function of the distance from one face, by the equations

$$U = 100x\ \ for\ 0 \le x \le 1, \qquad U = 100(2 - x),\ \ for\ \ 1 \le x \le 2$$

Find the temperatures as a function of x and t if both faces are maintained at $0^{0}C$. Where k for steel is

$$0.13cal/\sec.cm.^{0}C,\ c = 0.11cal/g.^{0}C \quad \rho = 7.8g/cm^{3}.$$

Solution: we subdivide the total thickness into an integral number of spaces. Let us use $\Delta x = 0.80cm,$

$$\frac{k\Delta t}{cp(\Delta x)^{2}} = \frac{1}{2}, \quad \Delta t = 0.206$$

Taking $\beta = 4\ ,\alpha = 16$ implies that $v = i - \frac{1}{4},\ z = i + \frac{1}{4}\ and\ \eta = \gamma = j + \frac{1}{16}.$

We take two interpolation points along space coordinate and one interpolation point along time coordinate. This Implies $g = 2,\ b = 1,\ and\ \ p = 3$. and for $i = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots,\ and\ \ j = \frac{1}{16}, \frac{1}{8}, \frac{3}{16}, \ldots,$ $h = -\frac{1}{4}, 0, \frac{1}{4}\quad and\ \ k = 0, \frac{1}{16}$, then the calculated temperatures are as shown in table 2.

Table 2: Calculated temperatures.

t	X=0	X=0.5	X=1.0	X=1.5	X=2.0	X=2.5	X=3.0	X=3.50	X=4
0	0	50	100	150	200	150	100	50	0
X=22.44	0	50	100	150	187.5	150	100	50	0
X=44.88	0	50	100	148.44	178.13	148.44	100	50	0
X=67.32	0	50	100	146.1	170.71	146.1	100	50	0
X=89.76	0	50	99.51	143.41	164.56	143.41	99.51	50	0
X=112.20	0	49.94	99.81	140.57	159.27	140.57	99.81	49.94	0
X=134.64	0	49.81	97.92	137.69	154.6	137.69	97.92	49.81	0

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